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# A NUMERICAL SOLUTION FOR AXIALLY SYMMETRICAL AND PLANE ELASTICITY PROBLEMS*

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**Abstract**—A finite difference method is presented for the solution of axially symmetrical and plane elasticity problems which makes efficient use of the high-speed digital computer. The method is ideally suited to the treatment of both simply and multiply connected regions with irregular boundary shapes, temperature distributions, body force distributions, and any physically admissible combination of boundary tractions and displacements. Numerical results are included showing close agreement with photoelastic results for a stress concentration problem.

## NOTATION

| | |
|---|---|
| $x, y$ | rectangular coordinates |
| $r, z$ | cylindrical coordinates |
| $s$ | boundary coordinate |
| $u, w$ | displacements corresponding to $r, z$ |
| $\sigma_r, \sigma_\theta, \sigma_z$ | normal stress components |
| $\tau_{rz}$ | shear stress component |
| $e$ | volume dilation |
| $\nu$ | Poisson's ratio |
| $\alpha$ | coefficient of thermal expansion |
| $G$ | shear modulus |
| $T$ | temperature |
| $R, Z$ | body force components per unit volume |
| $\bar{R}, \bar{Z}$ | boundary force components |
| $\bar{u}, \bar{w}$ | boundary displacement components |
| $U$ | total potential energy |
| $u_i, w_i$ | discrete values of $u, w$ at mesh point $i$ |
| $h_{ij}$ | mesh length between points $i$ and $j$ |
| $A$ | coefficient matrix |
| $D_j, B_j$ | matrices |
| $\zeta$ | column vector with components $u_i, w_i$ |
| $\mathbf{f}$ | load vector |
| $\omega$ | overrelaxation parameter |
| $\rho$ | spectral radius |

## 1. INTRODUCTION

IT is well known that finite difference methods provide reliable approximations for the solution of axially symmetrical and plane elasticity problems. However, for the more complex problems characterized by irregular boundary shapes and nonuniform loading conditions, the number of difference equations required for a reasonably accurate approximation can be so large that their solution is impractical without the aid of the digital computer.

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Both direct methods such as Gaussian elimination and indirect methods such as the iterative techniques described in [1] and [2] can be used to solve very large sets of equations provided the matrix of coefficients has the required properties. The relative advantages of direct and indirect methods depend on the properties of the equations to be solved as well as the specific methods being used. A comparison between an iterative and direct method for one system of equations is given in [3]. Although direct methods may be very efficient under certain conditions, they may require excessively large computer storage when the number of mesh lines in both coordinate directions is large. Furthermore, direct methods do not seem to be competitive with overrelaxation when applied to difference equations arising from systems of second order partial differential equations [3]. Thus, an iterative technique which makes efficient use of the digital computer was described in [4] for the solution of a limited class of plane elasticity problems. The difference equations were derived in such a manner that convergence of the iterative scheme is assured. It is the purpose of this paper to describe a similar method for the numerical solution of axially symmetrical problems. The new method is also applicable to a broader class of plane problems.

The axially symmetrical problem has been formulated in a number of different ways by introducing functions such as the biharmonic Love function, the two Southwell functions, and various pairs of displacement functions. Although these formulations may be convenient for the application of classical solution methods, they seem to offer no advantage when approximating by finite differences. In general it is difficult if not impossible to derive difference equations with the desired properties. However, such difference equations can be obtained easily using the more basic formulation in terms of displacement components. Solving directly for displacements, the treatment of boundary conditions on either displacements or tractions is greatly simplified, and multiply connected regions can be handled naturally while avoiding problems of uniqueness and single-valuedness. White [5] has already demonstrated the use of the displacement formulation for plane problems.

For the derivation of difference equations, the boundary of either the axially symmetrical ($rz$ plane) or plane region is approximated by a finite number of straight line segments, i.e. a polygonal shape. A rectangular mesh, which may be nonuniform, is imposed in such a way as to facilitate treatment of curved boundaries. The differential equations governing the two components of displacement are replaced by difference equations derived using a variational method based on minimization of total potential energy. It is shown that the resulting matrix of coefficients of the difference equations is symmetric, two-cyclic and positive semidefinite. A convergent iterative technique using the block overrelaxation method is described for the solution of the difference equations.

This method, which makes efficient use of the digital computer, can be used to solve many complex problems which could not be solved practically by classical methods. The applicability of the method is demonstrated by a comparison with photoelastic results for a stress concentration problem.

It should be noted that the difference equations obtained here are closely related to equations obtained by the finite element method which has recently been extended to the analysis of axially symmetrical bodies by Clough and Rashid [6]. As a matter of fact, using a modified form of the variational principal and consistent difference approximations, the same equations could be obtained by either method. The convergent iterative technique described here would be equally useful for the solution of equations obtained by the finite element method for two-dimensional elasticity problems.

## 2. THE BOUNDARY VALUE PROBLEM

Consider a body of revolution deformed symmetrically with respect to the axis of revolution by a temperature distribution $T(r, z)$ and a body force with components $R(r, z)$ and $Z(r, z)$ in the radial ($r$) and axial ($z$) directions. The equations of equilibrium in terms of displacement components $u$ and $w$ in the $r$ and $z$ directions, are

$$\nabla^2 u - \frac{u}{r^2} + \frac{1}{(1-2\nu)} \frac{\partial}{\partial r}[e - 2(1+\nu)\alpha T] + \frac{R}{G} = 0$$

$$\nabla^2 w + \frac{1}{(1-2\nu)} \frac{\partial}{\partial z}[e - 2(1+\nu)\alpha T] + \frac{Z}{G} = 0 \tag{1}$$

where

$\nu$ is Poisson's ratio

$G$ is the shear modulus

$\alpha$ is the coefficient of thermal expansion

$$e = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

Solving directly for displacements, compatibility of strains is identically satisfied.

The boundary of the region may be loaded by a stress with components $\bar{R}$ and $\bar{Z}$ in the $r$ and $z$ directions, by a specification of displacement components $\bar{u}$ and $\bar{w}$, or by a combination of stresses and displacements. The resulting boundary conditions may be described by

$$u = \bar{u} \quad \text{or} \quad 2G\left[\frac{\partial u}{\partial r} + \frac{\nu}{(1-2\nu)}e - \frac{(1+\nu)}{(1-2\nu)}\alpha T\right] l + G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) m = \bar{R}$$

and

$$w = \bar{w} \quad \text{or} \quad G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) l + 2G\left[\frac{\partial w}{\partial z} + \frac{\nu}{(1-2\nu)}e - \frac{(1+\nu)}{(1-2\nu)}\alpha T\right] m = \bar{Z} \tag{2}$$

where $l$ and $m$ are direction cosines defined in the usual manner.

The problems of plane strain and plane stress as defined in [7] are closely related to the axially symmetrical problem. Letting $r$ go to infinity ($1/r = 0$), terms containing the factor $1/r$ are eliminated, and the equilibrium equations (1) and boundary conditions (2) reduce to the equations for plane strain. Then replacing the material constants by

$$\nu = \frac{\nu_1}{(1+\nu_1)}; \qquad \alpha = \alpha_1 \frac{(1+\nu_1)}{(1+2\nu_1)}; \qquad G = G_1 \tag{3}$$

the equations for plane strain yield the equations for plane stress with $\nu_1$, $\alpha_1$, and $G_1$ the constants for plane stress. Since the finite difference equations for these problems are related in a similar manner, it is sufficient to consider here the derivation of difference

equations only for the axially symmetrical problem: the equations for the plane problems being obtained as described above.

The displacements in a body of revolution loaded symmetrically are obtained by solving (1) and (2). Then the stress components can be obtained from

$$\sigma_r = 2G\left[\frac{\partial u}{\partial r}+\frac{\nu}{(1-2\nu)}e-\frac{(1+\nu)}{(1-2\nu)}\alpha T\right]$$

$$\sigma_\theta = 2G\left[\frac{u}{r}+\frac{\nu}{(1-2\nu)}e-\frac{(1+\nu)}{(1-2\nu)}\alpha T\right]$$

$$\sigma_z = 2G\left[\frac{\partial w}{\partial z}+\frac{\nu}{(1-2\nu)}e-\frac{(1+\nu)}{(1-2\nu)}\alpha T\right]$$

$$\tau_{rz} = G\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right); \qquad \tau_{r\theta},\quad \tau_{z\theta} = 0. \tag{4}$$

with similar relations for the cases of plane strain and plane stress.

## 3. THE VARIATIONAL PROBLEM

Finite difference approximations could be derived for (1) and (2) using either Taylor's series expansions or the integration technique of [4]. An alternate approach is to approximate the strain energy, obtaining difference equations by the variational method. The strain energy approach is used here because of its mechanical simplicity. In addition, the rather formidable stress boundary conditions become "natural boundary conditions," i.e., they are included in the energy function and automatically satisfied by its minimization. It might be mentioned, however, that all three methods are closely related, leading to identical sets of equations when consistent difference approximations are used.

Let $U$ be the total potential energy of the system defined as the difference between the work of internal forces (strain energy) and external forces. Then according to the principle of virtual work, the displacements of the system due to the external forces are those which make

$$\delta U = 0 \tag{5}$$

for any virtual displacements from the equilibrium position. For a rotationally symmetric deformation, the total potential energy is expressed by

$$\begin{aligned} U = \frac{2\pi G}{(1-2\nu)}\iint\Bigg\{&\nu\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z}\right]^2+(1-2\nu)\left[\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{u}{r}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]\\ &+\tfrac{1}{2}(1-2\nu)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2-2(1+\nu)\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z}\right]\alpha T+3(1+\nu)\alpha^2T^2\Bigg\}r\,dr\,dz\\ &-2\pi\iint(Ru+Zw)r\,dr\,dz-2\pi\oint(\bar{R}u+\bar{Z}w)r\,ds \end{aligned} \tag{6}$$

where the line integral is taken around the boundary in the $rz$ plane. Similar expressions for plane strain and plane stress can be obtained by the method described previously.

The actual displacements $u$ and $w$ are those which minimize $U$ for any virtual displacements $\delta u$ and $\delta w$. It can easily be seen that (5) leads to the equilibrium equations (1) and

boundary conditions (2). Thus, the boundary value problem described by (1) and (2) can be replaced by the variational problem (5) where $U$ is given by (6).

## 4. DERIVATION OF THE DIFFERENCE EQUATIONS

The boundary of the region in either the $xy$ (plane strain and plane stress) or $rz$ (axially symmetrical) plane is approximated by a finite number of straight line segments. A rectangular mesh of horizontal and vertical lines is imposed on the region in such a manner that intersections of the rectangular mesh coincide with intersections of the boundary segments. In addition, a dual mesh is imposed on the region so that lines of the dual mesh fall halfway between lines of the mesh. Intersections of the rectangular mesh are called mesh points and the subregions bounded by boundary segments, mesh lines, and dual mesh lines are called cells. Each cell is either rectangular or right triangular and contains just one mesh point. This mesh layout is illustrated in Fig. 1 with boundary segments and the rectangular mesh shown as solid lines, the dual mesh as broken lines, and typical cells as shaded areas.

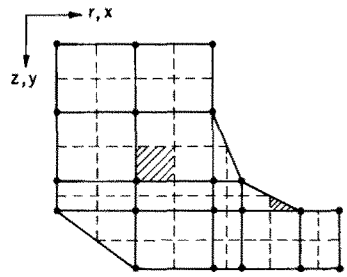


FIG. 1. Typical mesh layout.

The variational problem (5) in terms of the continuous variables $u(r, z)$ and $w(r, z)$ (or $u(x, y)$ and $v(x, y)$) is now replaced by a variational problem in terms of $u_i$ and $w_i$, the discrete values of $u(r, z)$ and $w(r, z)$ at mesh points $i$ ($i = 1, 2, \ldots, n$). This is done by approximating the total potential energy for each cell as a quadratic function of $u_i$ and $w_i$ at the three mesh points associated with each cell (Fig. 2). Summing over all cells, an approximation $U(u_i, w_i)$ is obtained for the total potential energy $U(u, w)$ of the system. Analogous to the variational problem (5), displacements $u_i$ and $w_i$ are sought which make $U(u_i, w_i)$ stationary. It can be shown that this is also a minimization problem.

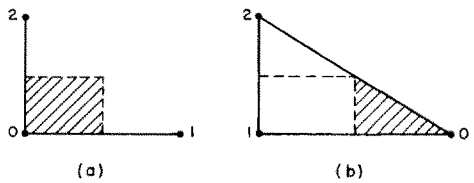


FIG. 2. Typical mesh cells.

An approximation for the potential energy (6) is obtained by taking all functions and their derivatives constant over each cell. Considering a typical rectangular cell, Fig. 2(a),

functions are approximated by

$$\frac{u}{r} \doteq \frac{u_0}{r_0}; \qquad T \doteq T_0; \qquad R \doteq R_0; \qquad \text{etc.} \tag{7}$$

while derivatives of functions are approximated by the differences

$$\frac{\partial u}{\partial r} \doteq \frac{(u_1-u_0)}{h_{01}}; \qquad \frac{1}{r}\frac{\partial}{\partial r}(ru) \doteq \frac{(r_1u_1-r_0u_0)}{r_0h_{01}}; \qquad \frac{\partial u}{\partial z} \doteq \frac{(u_0-u_2)}{h_{02}}; \qquad \text{etc.*} \tag{8}$$

where $h_{ij}$ is the mesh length between points $i$ and $j$. The approximation of (6) for the rectangular cell 021 then becomes

$$\begin{aligned} U_{021} = \frac{2\pi G}{(1-2\nu)}\Bigg\{ & \nu\left[\frac{(r_1u_1-r_0u_0)}{r_0h_{01}}+\frac{(w_0-w_2)}{h_{02}}\right]^2+(1-2\nu)\left[\frac{(u_1-u_0)^2}{h_{01}^2}+\frac{u_0^2}{r_0^2}+\frac{(w_0-w_2)^2}{h_{02}^2}\right] \\ & +\frac{(1-2\nu)}{2}\left[\frac{(u_0-u_2)}{h_{02}}+\frac{(w_1-w_0)}{h_{01}}\right]^2+3(1+\nu)\alpha^2T_0^2 \\ & -2(1+\nu)\left[\frac{(r_1u_1-r_0u_0)}{r_0h_{01}}+\frac{(w_0-w_2)}{h_{02}}\right]\alpha T_0\Bigg\} r_0\frac{h_{01}h_{02}}{4} \\ & -\frac{\pi}{2}(R_0u_0+Z_0w_0)r_0h_{01}h_{02}-2\pi\int_b(\bar{R}u+\bar{Z}w)r\,\mathrm{d}s \end{aligned} \tag{9}$$

where $\int_b$ is taken only over sides of the cell coinciding with the boundary of the region. The boundary forces $\bar{R}$ and $\bar{Z}$ are the so called "natural boundary conditions". They are automatically satisfied by the minimization of $U$. If, however, "constraints" are imposed, i.e., boundary displacements specified, the forces associated with the known boundary displacements do no work during a virtual displacement, and so do not enter the problem. The known values of $u$ and $w$ at boundary mesh points are substituted into the energy approximation, and the corresponding line integrals omitted. Thus, at each boundary point we may specify $\bar{R}$ or $u$ and $\bar{Z}$ or $w$. For example, if 02 is an interior segment and 01 a boundary segment with $u$ and $\bar{Z}$ known, the line integral in (9) becomes

$$\int_b \bar{Z}wr\,\mathrm{d}s \doteq \bar{Z}_{01}w_0r_0\frac{h_{01}}{2}$$

where $\bar{Z}_{01}$ is the boundary force taken constant on 01, and $u_0$ in (9) must be replaced by its known value $\bar{u}_0$.

For cells adjacent to the line of axial symmetry, $r = 0$, the stress expressions (4) and energy expression (6) must be modified. At $r = 0$, $u = 0$, $\varepsilon_\theta = \varepsilon_r = \partial u/\partial r$, and the stress components (4) become

$$\begin{aligned} \sigma_r = \sigma_\theta &= 2G\left[\frac{\partial u}{\partial r}+\frac{\nu}{(1-2\nu)}e-\frac{(1+\nu)}{(1-2\nu)}\alpha T\right] \\ \sigma_z &= 2G\left[\frac{\partial w}{\partial z}+\frac{\nu}{(1-2\nu)}e-\frac{(1+\nu)}{(1-2\nu)}\alpha T\right] \\ \tau_{rz} = \tau_{r\theta} &= \tau_{z\theta} = 0 \end{aligned} \tag{10}$$

* It might be observed that if $(1/r)(\partial/\partial r)(ru)$ had been written as $\partial u/\partial r + u/r$, a different approximation would have been obtained. But the order of accuracy is the same, and the form used here reduces the storage requirement for the computer program.

where

$$e = 2\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}.$$

The corresponding expression for total potential energy is

$$U = \frac{2\pi G}{(1-2\nu)} \iint \left\{ \nu \left( 2\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \right)^2 + (1-2\nu)\left[ 2\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \right] - 2(1+\nu)\left( 2\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \right)\alpha T + 3(1+\nu)\alpha^2 T^2 \right\} r \, dr \, dz$$

$$- 2\pi \iint (Ru + Zw) r \, dr \, dz - 2\pi \int (\bar{R}u + \bar{Z}w) r \, ds. \tag{11}$$

Thus, if segment 02 in Fig. 2(a) coincides with the line of axial symmetry, the energy approximation for the cell 021 becomes

$$U_{021} = 0 \tag{12}$$

as it does for all other cells adjacent to the line $r = 0$.

The approximation of the total potential energy for a triangular cell, Fig. 2(b), proceeds in a similar fashion. Functions are approximated as in (7) while derivatives of functions are approximated by

$$\frac{\partial u}{\partial r} \doteqdot \frac{(u_0 - u_1)}{h_{01}}; \qquad \frac{1}{r}\frac{\partial}{\partial r}(ru) \doteqdot \frac{(r_0 u_0 - r_1 u_1)}{r_0 h_{01}}; \qquad \frac{\partial u}{\partial z} \doteqdot \frac{(u_1 - u_2)}{h_{12}}; \qquad \text{etc.} \tag{13}$$

It might be observed that the same approximations (13) are used for all three cells shown in Fig. 2(b).

With an approximation such as (9) for each cell, the total potential energy of the system is expressed by

$$U = \sum U_{ijk} \tag{14}$$

where the summation is over all cells. Difference equations relating the $p$ unknown $u_i$ and $q$ unknown $w_i$ are then obtained from the conditions for $U$ stationary

$$\frac{\partial U}{\partial u_i} = 0 \qquad i = 1, 2, \ldots, p \qquad p \leq n$$

$$\frac{\partial U}{\partial w_i} = 0 \qquad i = 1, 2, \ldots, q \qquad q \leq n. \tag{15}$$

The equation for a typical interior point 0, Fig. 3, obtained from the first of (15), $\partial U/\partial u_0 = 0$, is

$$\frac{r_0}{2}(h_{01} + h_{03}) \sum_{i=2,4} (u_0 - u_i)\frac{1}{h_{0i}} + \frac{(h_{02} + h_{04})}{(1-2\nu)} \sum_{i=1,3} \left[ \left( 1 - 2\nu + \nu\frac{r_0}{r_i} \right) u_0 - (1-\nu)u_i \right] \frac{(r_0 + r_i)}{2h_{0i}}$$

$$+ \frac{2u_0}{r_0} A_0 + \frac{r_0}{4(1-2\nu)}(w_7 + w_5 - w_6 - w_8) = -\frac{(1+\nu)\alpha r_0}{2(1-2\nu)}(h_{02} + h_{04})(T_1 - T_3) + \frac{r_0 R_0 A_0}{G} \tag{16}$$

where

$$A_0 = \tfrac{1}{4}(h_{01}h_{02} + h_{02}h_{03} + h_{03}h_{04} + h_{04}h_{01}).$$

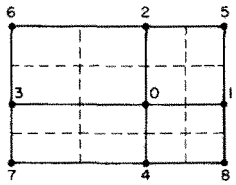


FIG. 3. Typical interior mesh point.

It is easily seen that (16) represents a finite difference approximation to the first of equations (1) which is of $O(h)$ accuracy when the mesh spacings are not equal, and $O(h^2)$ accuracy when the mesh spacings are equal. The same is true for the approximation to the second of equations (1).

Numbering the mesh points in a systematic way, the $p+q$ linear difference equations (15) can be written in the matrix form

$$A\zeta = \mathbf{f} \tag{17}$$

where $A$ is a $(p+q)\times(p+q)$ coefficient matrix, $\zeta$ is a column vector with components $u_i$ and $w_i$, and $\mathbf{f}$ is a known column vector representing the external loads. It should be noted, however, that the solution for $u_i$ and $w_i$ will not be unique, i.e., $A$ will be singular, unless $w_i$ is specified for at least one point ($q \leq n-1$). Physically, this serves to locate the axially symmetrical body in space. Similarly, for plane problems, the solution will not be unique unless three components of displacement are known: $u_i$ at one point and $v_i$ at two points or $v_i$ at one point and $u_i$ at two points. A more detailed account of the derivation of difference equations (17) can be found in [8].

Although the problem of spatially uniform material properties has been considered, it is apparent that the problem of spatially variable material properties can be treated just as easily. Since the integrations in (6) are carried out cellwise, the material properties can be allowed to vary from one mesh rectangle to the next.

## 5. ITERATIVE SOLUTION OF THE DIFFERENCE EQUATIONS

When the difference approximations of Section 4 are used to solve complex problems, a relatively fine mesh is required. As a result, the number of unknowns in (17) may be very large, perhaps several thousand. An efficient iterative scheme for the system (17) is that of successive line overrelaxation. Convergence is assured if the coefficient matrix $A$ is symmetric and positive semidefinite as shown in the Appendix. Conditions under which $A$ is nonsingular are also given in the Appendix. Rate of convergence is increased by making $A$ two-cyclic [2].

To apply the method of successive line overrelaxation, the unknowns and equations are ordered in the following manner. The columns of mesh points are numbered from left to right, the left most column being column 1. On a given column of mesh points, the points are numbered from top to bottom. The displacements $u$, $w$ at a point $i$ are arranged in the

order $u_i, w_i$ if both $u_i$ and $w_i$ are unknown. If one of $u$ or $w$ is known at the mesh point $i$, the ordering of the unknowns at $i$ is unambiguous. This ordering of mesh points, and unknowns at a mesh point, combines to give an ordering of the unknowns of the problem.

The unknowns are partitioned into blocks. The $j$-th block of unknowns, $\zeta_j$, corresponds to the displacements at mesh points on the $j$-th column. The coefficient matrix $A$ is partitioned in a manner consistent with the partitioning of the unknowns. Since the displacements in any column are coupled only to the displacements in adjacent columns, the partitioned matrix $A$ takes the block tridiagonal form

$$A = \begin{bmatrix} D_1 & -B_1 & & & \\ -B_1^T & D_2 & -B_2 & & \\ & -B_2^T & D_2 & \ddots & \\ & & \ddots & \ddots & -B_{J-1} \\ & & & -B_{J-1}^T & D_J \end{bmatrix}$$

where $J$ is the number of columns of mesh points. With this partitioning the difference equations (17) become

$$\begin{cases} D_1\zeta_1 - B_1\zeta_2 = \mathbf{f}_1, \\ D_j\zeta_j - B_{j-1}^T\zeta_{j-1} - B_j\zeta_{j+1} = \mathbf{f}_j, \qquad 1 < j < J, \\ D_J\zeta_J - B_{J-1}^T\zeta_{J-1} = \mathbf{f}_J. \end{cases} \tag{18}$$

The diagonal blocks $D_j$ are symmetric, 7—diagonal matrices; i.e., if $D_j = (d_{kl}^{(j)})$, then $d_{kl}^{(j)} = 0$ for $|k-l| > 3$. It will be proved in the Appendix that $D_j$ is positive definite, even in the case when $A$ is singular. Hence matrix equations of the form

$$D_j\zeta_j = \mathbf{g}_j, \tag{19}$$

where $\mathbf{g}_j$ is given, can be solved directly by means of Gaussian elimination with little accumulation of rounding errors [9].

The fact that (19) can be solved efficiently allows the use of successive line overrelaxation for the solution of (18). The iterative procedure is

$$D_j\zeta_j^{(m+1)} = \omega\{B_{j-1}^T\zeta_{j-1}^{(m+1)} + B_j\zeta_j^{(m)} + \mathbf{f}_j\} + (1-\omega)\zeta_j^{(m)}, \qquad m = 1, 2, \ldots \tag{20}$$

An initial guess $\zeta^{(0)}$ is required to start the iterations, and (20) is applied in the order $j = 1, 2, \ldots, J$.

From the usual theory of overrelaxation, if $A$ is nonsingular, the iterations (20) will converge for any $\omega$, $0 < \omega < 2$, and for any initial guess $\zeta^{(0)}$. The convergence rate is largest if $\omega$ equals the optimal value

$$\omega_b = \frac{2}{1+\sqrt{(1-\rho^2)}}, \tag{21}$$

where $\rho$ is the largest absolute value of the eigenvalues of an associated iteration matrix.

The same results hold if $A$ is singular. In this case there will be some null vectors of $A$, vectors $\boldsymbol{\mu}$ such that $A\boldsymbol{\mu} = 0$. The system of equations (17) then has a solution $\zeta$ if and only if

$$A\boldsymbol{\mu} = 0 \text{ implies } \boldsymbol{\mu}^T\mathbf{f} = 0. \tag{22}$$

Physically this occurs when the body is in a state of static equilibrium under the system of applied loads. If this condition is satisfied, it follows from the results of [10] that, in the singular case also, the iterations (20) will converge for $0 < \omega < 2$. A simple modification of the theory of overrelaxation shows that (21) gives the optimal value of $\omega$, provided that $\rho$ is the largest absolute value of the eigenvalues not equal to 1 of an associated iteration matrix.

The convergence rate of (20) is highly dependent on an accurate estimate of $\rho$. A method can be given for obtaining $\rho$ which requires an auxiliary iterative calculation. Convergence of the iterations is accelerated using Chebyshev polynomials. The time spent in the calculation of an accurate value of $\rho$ is more than made up in the more rapid convergence of (20). Details for implementing these iterative techniques are given in [8].

## 6. APPROXIMATION OF STRESS COMPONENTS

Stresses are expressed in terms of displacements by (4) except on the line of axial symmetry, $r = 0$, where they are given by (10). With a solution $\zeta$ for the displacement components $u_i$ and $w_i$, approximations for the stress components can be obtained using the same approximations as used in the energy integrals. Since functions and their derivatives are taken uniform over each cell, stress components are likewise uniform. A stress component at a mesh point is obtained by taking an area weighted average of the stress component over all cells which include that mesh point.

For example, at an interior mesh point 0, Fig. 3, there are four cells containing point 0. The difference approximation for $\sigma_r$ in cell 021 is

$$(\sigma_r)_{021} = 2G\left\{\frac{(u_1-u_0)}{h_{01}}+\frac{\nu}{(1-2\nu)}\left[\frac{(r_1u_1-r_0u_0)}{r_0h_{01}}+\frac{(w_0-w_2)}{h_{02}}\right]-\frac{(1+\nu)}{(1-2\nu)}\alpha T_0\right\}.$$

Taking the area weighted average over the four cells, the approximation for $\sigma_r$ at point 0 is

$$(\sigma_r)_0 = 2G\left\{\frac{(u_1-u_3)}{(h_{01}+h_{03})}+\frac{\nu}{(1-2\nu)}\left[\frac{(r_1u_1-r_3u_3)}{r_0(h_{01}+h_{03})}+\frac{(w_4-w_2)}{(h_{02}+h_{04})}\right]-\frac{(1+\nu)}{(1-2\nu)}\alpha T_0\right\}.$$

## 7. COMPARISON WITH EXPERIMENTAL RESULTS

Since the accuracy of finite difference methods is well known, there is little need for comparison with exact solutions to problems with regular boundary shapes and simple loading conditions. Of greater interest is the ability of the method described here to handle the more difficult problems in irregularly shaped regions which cannot be solved practically by classical analytical methods. To demonstrate the capability of this method it has been used to determine the stress distribution in an internally pressurized cylinder with an external circumferential fillet as shown in Fig. 4. The resulting stress concentrations at the fillet are compared to those obtained by Leven [11] using the photoelastic technique. It

should be noted that the fillet radius (0·05 in.) is small compared to the other dimensions and thus provides a severe test of the numerical method.

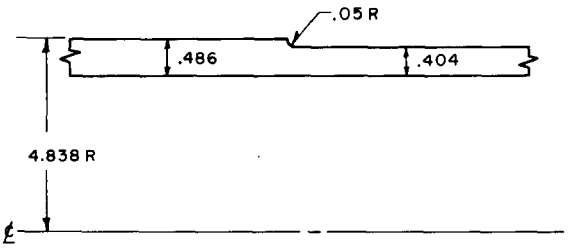


FIG. 4. Internally pressurized cylinder with exterior fillet.

For the finite difference solution, the cylinder was truncated at a distance of about 1·3 times the radius on either side of the fillet where the effects of the discontinuity have become negligible. Stresses of the exact theory for a cylinder of uniform thickness were applied at the ends of the truncated cylinder. A nonuniform mesh was used containing 525 interior and boundary mesh points, with a concentration of points in the vicinity of the fillet as shown in Fig. 5. The numerical solution, obtained using the DUZ-1 computer program [8], required about 11 min on the Philco-2000 computer.

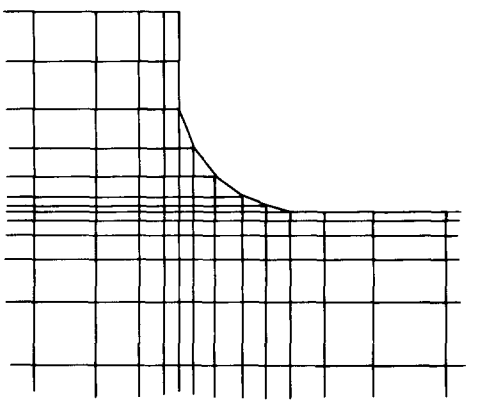

FIG. 5. Mesh layout in the vicinity of the fillet.

Meridional and hoop stress distributions in the vicinity of the fillet, obtained by both the numerical and experimental techniques, are plotted for comparison in Fig. 6. It can be seen that the meridional stress ratios agree well, although a slight difference in location of the peak may be indicated. Agreement between the hoop stress ratios is less satisfactory, however, probably well within the range of experimental accuracy. According to Leven the hoop stress in the vicinity of the maximum is most difficult to measure and may well be off by 10 per cent or so. On the other hand, the meridional stress can be measured quite accurately, so the close agreement obtained there is reassuring to both the numerical analyst and the experimentalist.

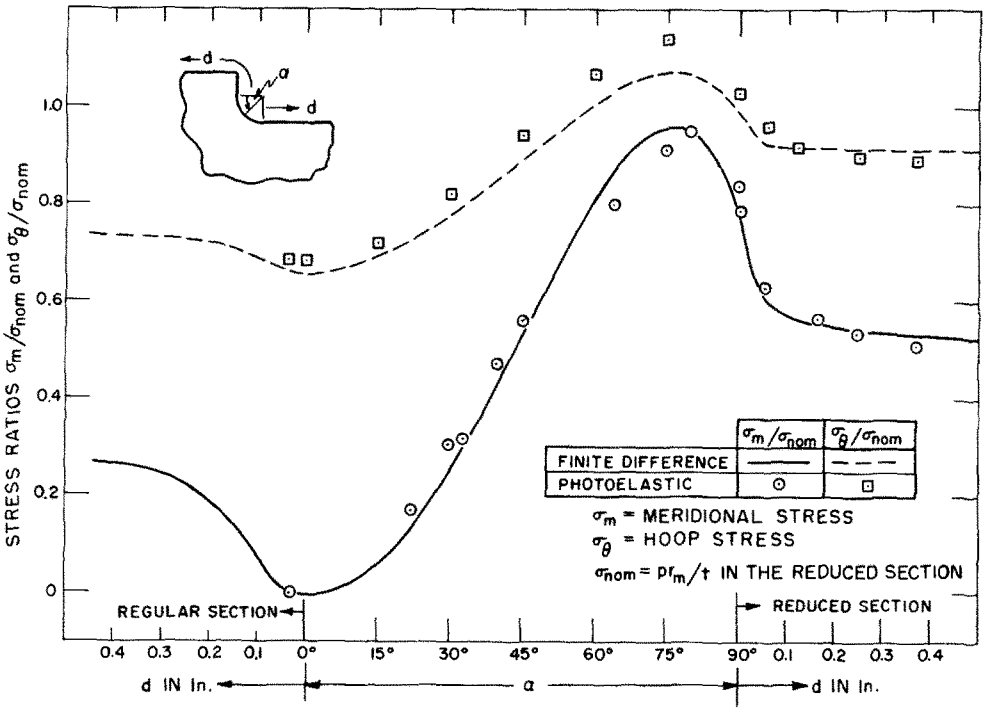


FIG. 6. Comparison of numerical and experimental results in the vicinity of the fillet.

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## APPENDIX

### *Properties of the Matrix A*

In defining the matrix $A$ of (17) the unknown displacements $u_i$ and $w_i$ have been arranged as the components of a column vector $\zeta$. It is convenient to introduce the vector space $V$

of all such column vectors. Then the approximate total potential energy, $U$, is a numerical function of the vectors $\zeta$ of $V$ which is a quadratic polynomial in the components of $\zeta$. If this polynomial is decomposed into its quadratic, linear, and constant parts, there results a representation of $U$ as

$$U(\zeta) = Q(\zeta) - \mathbf{f}^T\zeta + \zeta_0, \tag{23}$$

where $Q(\zeta)$ is a quadratic form on $V$, and $\mathbf{f}$ and $\zeta_0$ are fixed vectors in $V$. The difference equations (15) now become

$$\frac{\partial Q(\zeta)}{\partial \zeta_k} = f_k \qquad k = 1, 2, \ldots, p+q,$$

and the matrix $A$ is a symmetric matrix whose entries are $\frac{1}{2}$ the coefficients of the quadratic form $Q$. Furthermore, $Q(\zeta)$ may be written

$$Q(\zeta) = \zeta^T A \zeta. \tag{24}$$

Now if $\mathbf{f} = \zeta_0 = 0$, one has from (23), (24), and the various expressions for the approximate total potential energy $U$ [e.g. (9)], that

$$Q(\zeta) = \zeta^T A \zeta \geq 0. \tag{25}$$

Hence we have established the following result.

LEMMA 1. *$A$ is a symmetric, positive semidefinite matrix.*

The next task is to develop the properties of the matrices $D_j$ which form the diagonal blocks of $A$ when it is partitioned into a block tridiagonal matrix in the manner described above. Each $D_j$ is symmetric and positive semidefinite, and may be described in terms of its action on a vector as follows. Let $\zeta^{(j)}$ be a vector of displacements such that all the displacements vanish except those on column $j$. The vector $A\zeta^{(j)}$ then has all components 0 except possibly those corresponding to displacements on columns $j-1$, $j$, and $j+1$. If $A\zeta^{(j)}$ is modified by requiring the displacements on columns $j-1$ and $j+1$ to vanish, a new vector is obtained which may be identified with the vector $D_j\zeta^{(j)}$. One has

$$Q(\zeta^{(j)}) = \zeta^{(j)T} A \zeta^{(j)} = \zeta^{(j)T} D_j \zeta^{(j)}.$$

To show that $D_j$ is positive definite, it must be shown that $Q(\zeta^{(j)}) = 0$ only when $\zeta^{(j)} = 0$. If in Fig. 2(a), the mesh points 0 and 2 lie on column $j$, and if in (9) the deflections are components of a vector $\zeta^{(j)}$, so that $u_1 = w_1 = 0$, then the contribution to $Q$ coming from the rectangular cell 021 is

$$Q_{021} = \frac{2\pi G}{(1-2\nu)}\left\{\nu\left[\frac{-r_0 u_0}{r_0 h_{01}} + \frac{(w_0 - w_2)}{h_{02}}\right]^2 + (1-2\nu)\left[\frac{u_0^2}{h_{01}^2} + \frac{u_0^2}{r_0^2} + \frac{(w_0-w_2)^2}{h_{02}^2}\right] + \frac{(1-2\nu)}{2}\left[\frac{(u_0-u_2)}{h_{02}} - \frac{w_0}{h_{01}}\right]^2\right\}.$$

If $Q = 0$, then $Q_{021} = 0$ and it is readily seen that $u_0 = u_2 = w_0 = w_2 = 0$. Thus:

LEMMA 2. *The diagonal blocks $D_j$ in the tridiagonal partitioning of $A$ are symmetric, nonsingular, positive definite matrices.*

It remains to consider when the matrix $A$ is singular. Since $A$ is symmetric, this can happen only when $Q(\zeta) = 0$ for a non-zero vector $\zeta$. The quadratic form $Q(\zeta)$ is the sum

of a number of non-negative forms; a typical summand is the form $Q_{021}(\zeta)$ obtained from $U_{021}$ of (9) by deleting the terms which are not quadratic in the deflections. If $Q(\zeta) = 0$, then $Q_{021}(\zeta) = 0$, from which it follows that $u_0 = 0$ and $w_0 = w_1 = w_2$. From this we immediately obtain:

LEMMA 3. *A has at most one linearly independent null vector, given by the deflections* $u_i \equiv 0, w_i \equiv$ *constant. If the boundary conditions include a specification of w at any boundary point, then A is nonsingular.*

Suppose $A$ is singular and let $\boldsymbol{\mu}$ be the null vector of $A$. (According to Lemma 3, $\boldsymbol{\mu}$ is unique up to a constant multiple.) Equation (17) will have a solution if and only if

$$\boldsymbol{\mu}^T \mathbf{f} = 0. \tag{26}$$

The vector $\mathbf{f}$ is composed of two parts. One part comes from the temperature gradients, and automatically satisfies (26). The other part comes from the body and boundary forces, and (26) asserts that the $z$ components of these forces must be in equilibrium. Thus the orthogonality condition (26), mathematically required for (17) to have a solution, corresponds to the physical requirement of static equilibrium. If (26) is satisfied and (17) has a solution, it has an infinity of solutions any two of which differ by $\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is the null vector described in Lemma 3.

In the cases of plane stress and plane strain, the total potential energy $U_{021}$ of (9) no longer contains the term

$$(1-2\nu)\frac{u_0^2}{r_0^2}.$$

This causes the matrix $A$ to have up to three linearly independent null vectors. Physically, these vectors correspond to translations in the $x$ and $y$ directions, and to a rotation. Further details are given in [8].

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**Résumé**—Une méthode à différence finie est présentée pour la solution de problèmes d'élasticité plane et à symétrie axiale qui utilise de la façon la plus efficace la machine à calculer digitale à grande vitesse. La méthode convient idéalement au traitement de régions à liaison simple et multiple à formes aux limites irrégulières, distributions de température irrégulières, distributions de force du corps irrégulières et toute combinaison physiquement admissible des déplacements et tractions aux limites. Des résultats numériques sont inclus qui présentent des conclusions très proches de celles des résultats photoélastiques pour un problème de concentration des tensions.

**Zusammenfassung**—Eine endliche Differenzmethode zur Lösung axialsymmetrischer und eben-elastischer Probleme wird gegeben, die einen Hochgeschwindigkeits-Digitalrechner auf leistungsfähige Art verwendet. Die Methode eignet sich insbesondere zur Behandlung einfacher sowie mehrfach verbundener Regionen mit unregelmässigen Grenzformen, Temperaturverteilungen, Körperkraftverteilungen, sowie aller physikalischer zulässigen Kombinationen von Grenzkräften une Verschiebungen. Numerische Resultate werden gegeben die eine gute Übereinstimmung mit photoelastischen Resultaten des Spannungs-Konzentrations Problemes geben.

**Абстракт**—Излагается метод конечных разностей для решения осесимметричных и плоских задач упругости, который оказывается очень эффективным при использовании быстродействующей цифровой вычислительной машины. Метод идеально удовлетворяет требовании обработки равно как простых так и многосвязных областей с нерегулярной формой контура, распределением температуры, распределением массовых сил и произвольной физически допускаемой комбинацией сил сцепления и перемещений на контуре. Приводятся численные расчеты, которые указывают строгое согласие с результатами полученными методом фотоупругости для задачи, касающейся концентрации напряжений.